ON CERTAIN DETERMINANT FORMS AND THEIR APPLICATIONS.

(SECOND PAPER.—*Continued.*)

III. CERTAIN TRANSCENDENTAL FUNCTION SERIES.

27. Hagen, in his Synopsis, introduces the subject of the Expansion of Functions in Series with the remark that "Jacob Bernoulli vergleicht in seinem Tractatus de Seriebus (Basil, 1713, Praefatio) die Reihenentwickelung mit einem Anker, der in hoffnungslosen Problemen, wo alle anderen Kräfte des menschlichen Scharfsinns Schiffbruch leiden, das letzte Zufluchtsmittel bilde." Yet it was after this that die Reihenentwickelung instead of an anchor proved a dangerous breaker upon which so many German mathematicians drifted, and it was not until Cauchy and Abel took hold of the subject, a full century later, that the theory of the expansion of functions in infinite series was placed upon a sound scientific basis, through the criteria then offered for the determination of the convergency of a given series.

No universal method for the *deduction* of a general expansion of a function in infinite series in terms of given functions has ever been designed, the discovery of particular expansions of an arbitrary function in series of functions of specified form has been reached in several ways. First; by the method of undetermined coefficients which assumes the expansion and determines the values which the unknown coefficients must have if the series does exist and then seeks to prove its existence or the arithmetical equivalence of the result so obtained with the given function. In the case of the simpler functions these undetermined coefficients may be evaluated through the process of repeated differentiation and in more complex forms by an ingenious application of definite integration, in which case the series is multiplied through by some function such that when the series is integrated between certain limits, all terms vanish save the one involving the coefficient desired and a definite integral of the product of the function to be expanded and the multiplier-function. Examples of this method are recognized in the usual treatment of the integer-power series and expansions in trigonometrical series, Legendre's and Bessel's functions. Second; by a repeated application of integration by parts certain series may be actually developed, the terminal term being expressed as a definite integral. This rigorous method of deducing series is so much more satisfactory than the preceding that it has gone far towards removing the subject of the expansion of functions in series from the domain of the

Differential to that of the Integral Calculus. Indeed if we may go further and include under this head the magnificent theorems of Cauchy respecting definite integration of complex functions, upon which the most rigorous deduction of series which we possess is based, it may be said that the Integral Calculus is now the true province of this subject. Third; by the method of generating functions, of which extensive use was made by the older analysts, well illustrated by a method due to Abel in his Œuvres Completes, also in Lacroix, Diff. and Int. Cal. III, 322. A fourth method of analysis should be added to these, i. e. that of the Calculus of Operations, illustrated and developed by Dr. McClintock in his Essay on the Calculus of Enlargement, and in Boole's Finite Differences.

In the absence of a general method of deducing series and the dependence of their discovery upon, as it were, methods of chance, ingenuity and skill, so great was the importance attached to the invention of one of these powerful instruments of analysis, that no more fitting honor could be bestowed upon the author than to place his name with his series among the imperishable theorems of mathematical analysis; thus, with all the magnificent creations of Newton's mind, his tomb in Westminster Abbey bears the expansion of the binomial.

The subject of the expansion of an arbitrary function in terms of transcendental functions was perhaps first touched successfully by Lagrange when he effected the expansion in terms of sines (Ancien Mémoires de l'Academie de Turin), which was followed by Fourier's celebrated theorem regarding the expansion of a complex harmonic function (Théorie Analytique de la Chaleur. Paris, 1822). Since then we have had several remarkable expansions in this subject, which constitutes in itself an important and interesting branch of analysis, such as the expansions of arbitrary functions in terms of Legendre's, Laplace's, and Bessel's functions, among which an instructive illustration is the expansion of Schlömilch in terms of Besselian functions (Todhunter's Functions, p. 336); not to mention the elegant work which has recently been done in this direction in regard to elliptic functions.

There is perhaps no better presentation of Fourier's Theorem than that given by Thompson and Tait (Nat. Philos. I. 54), where it is said of this theorem that it "is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance." It may be said in general of this whole class of series that the subject is one of the utmost

importance in the higher parts of mathematical physics, so that any investigation which it seems may throw light on the matter is worthy of being made even though the results prove nugatory; for it may suggest some more powerful method of analysis whereby desired forms may be obtained. "I am not one of those," says Sylvester, "who look upon Analysis as only valuable for the positive results to which it leads, and who regard proofs as almost a superfluity, thinking it sufficient that mathematical formulæ should be obtained, no matter how, and duly entered on a register.

"I look upon Mathematics not merely as a language, an art, and a science, but also a branch of Philosophy, and regard the forms of reasoning which it embodies and enshrines as among the most valuable possessions of the human mind. Add to this that it is scarcely possible that a well-reasoned mathematical proof shall not contain within itself subordinate theorems—germs of thought of intrinsic value and capable of extended application."

In approaching the vast problem of the expansion of any arbitrary function whatever in terms of given functions, the general problem bears insolubility on its face, save only under certain conditions and for certain values of the quantities which the functions contain. Even though the deduced series fulfil the conditions for convergency, it will be an arithmetical equality with the function to be expanded for general values of the variable, only when that function is such that it enjoys the same characteristic properties as do the functions in terms of which it is expressed, otherwise only between certain definite values of the variable does this equality exist.

I do not profess to offer here a discussion of the general problem, but merely to take up one particular case only of a particular class, and in this class to give examples of the deduction of a number of series in order to illustrate a method. Of the series presented I do not intend to attempt here a discussion as to their limits of applicability, for from the general nature of these series we would be frequently led altogether beyond the design of the object immediately in view.

28. Heretofore attention has been confined to the application of the composite towards effecting the expansion of an arbitrary function in terms of that class of functions which were such that their derivatives on one side of the diagonal of the composite vanished, through the operation of differentiation or through the substitution after differentiation of a specific arbitrary constant for the variable, thus yielding unity for a body-determinant. This gives rise to two forms of expansion, one of which is arranged according to the variable functions the coefficients of whose terms are dependent upon the derivative forms of the given function to be developed; while the other is arranged according to the derivative forms of this function, and whose coefficients are

wholly independent of the *form* of the function to be expanded. In the one case or the other, depending upon whether the elements above or below the diagonal vanish, the coefficients may be expressed by recurrence formulæ.

We now propose the study of the expansion of an arbitrary function in terms of a certain class of transcendental functions which are such as yield a body-determinant which is a difference-product, giving examples of expansions in terms of functions which give the particular body-determinant

$$\zeta^{\frac{1}{2}}(1^2, 2^2, \ldots, n^2).$$

29. In order to illustrate the general method of procedure we consider the following general theorem :—

To expand an arbitrary function $f(x+h)$ in terms of the arbitrary functions

$$\frac{1}{a_r}\varphi(k+a_rbx),$$

in which k, a_r and b are arbitrary constants, r taking in succession the values 1, 2, 3, ...

Making the substitutions in the composite, putting $x=0$ after each differentiation, and factoring out the common factors, we have

$$\begin{vmatrix} f(x+h), & 1, & \frac{1}{a_1}\varphi(k+a_1bx), & \frac{1}{a_2}\varphi(k+a_2bx), & \ldots, & \frac{1}{a_n}\varphi(k+a_nbx), & \frac{1}{a_{n+1}}\varphi(k+a_{n+1}bx) \\ f(y+h), & 1, & \frac{1}{a_1}\varphi(k+a_1by), & \frac{1}{a_2}\varphi(k+a_2by), & \ldots, & \frac{1}{a_n}\varphi(k+a_nby), & \frac{1}{a_{n+1}}\varphi(k+a_{n+1}by) \\ \frac{1}{b}\frac{f'h}{\varphi'k}, & 0, & 1, & 1, & \ldots, & 1, & 1 \\ \frac{1}{b^2}\frac{f''h}{\varphi''k}, & 0, & a_1, & a_2, & \ldots, & a_n, & a_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{b^n}\frac{f^nh}{\varphi^nk}, & 0, & a_1^{n-1}, & a_2^{n-1}, & \ldots, & a_n^{n-1}, & a_{n+1}^{n-1} \\ \Phi(u), & 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0. \qquad (113)$$

Indicating the body-determinant by the symbol Δ, we have in (113)

$$\Delta \equiv \zeta^{\frac{1}{2}}(a_1, \ldots, a_n) \equiv \begin{vmatrix} 1, & 1, & \ldots, & 1 \\ a_1, & a_2, & \ldots, & a_n \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{n-1}, & a_2^{n-1}, & \ldots, & a_n^{n-1} \end{vmatrix}.$$

It is to be noticed, that while in (113) we put $x = 0$ after differentiation, we may, when actually applying the theorem, instead of zero, put for x, after differentiation, any constant which will cause the derivatives of the φ functions in that row to have the same value so that they may be factored from the body-determinant. Or, in general, substitute for the variable, after differentiation, any arbitraries, which will make the body-determinant a difference-product.

30. We expand (113) after the manner of (13), that is, with respect to the last row obtaining two determinants; the minor of the last element of the last row we expand with respect to its second column, thus obtaining again two determinants, which differ from each other only in that y in one replaces x in the other. We now divide the equality through by Δ, and calling its three terms Fx, Fy, and R, respectively, wherein R represents the term in $\Phi(u)$ and its minor divided by Δ, we write (113) thus:—

$$Fx = Fy + R. \tag{114}$$

We may expand Fx in two ways; either with respect to its first row according to the φ functions, or we may expand it according to the first column, arranging the series in this case according to the derivative forms of f.

Let $\Delta(r, p)$ be the determinant Δ deleted with respect to its rth column and pth row.

Let $\Delta_f(r)$ be the determinant Δ in which the rth column has been replaced by the column of derivative forms of f and φ,

$$\frac{1}{b}\frac{f'h}{\varphi'k}, \ldots, \frac{1}{b^n}\frac{f^nh}{\varphi^nk}.$$

Let $\Delta_x(p)$ be the determinant Δ in which the pth row has been replaced by the row of φ functions,

$$\frac{1}{a_1}\varphi(k + a_1bx), \ldots, \frac{1}{a_n}\varphi(k + a_nbx);$$

with a corresponding meaning attached to $\Delta_y(p)$.

Expanding Fx with respect to its first row, we have

$$Fx = f(x + h) - \sum_{r=1}^{r=n} \frac{\Delta_f(r)}{a_r\Delta}\varphi(k + a_rbx).$$

Since, if in this we change x into y we get Fy, we have for the expansion of (113) with respect to the φ functions,

$$f(x + h) - f(y + h) = \sum_{r=1}^{r=n} \frac{\Delta_f(r)}{a_r\Delta}[\varphi(k + a_rbx) - \varphi(k + a_rby)] + R. \tag{115}$$

Expanding $\Delta_f(r)$ with respect to its rth column, we have

$$(-1)^{r+1}\frac{\Delta_f(r)}{\Delta} = \frac{1}{b}\frac{\Delta(r,1)}{\Delta}\frac{f'h}{\varphi'k} - \frac{1}{b^2}\frac{\Delta(r,2)}{\Delta}\frac{f''h}{\varphi''k} + \ldots + (-1)^{n+1}\frac{1}{b^n}\frac{\Delta(r,n)}{\Delta}\frac{f^n h}{\varphi^n k},$$

$$= \sum_{p=1}^{p=n}\frac{(-1)^{p+1}}{b^p}\frac{\Delta(r,p)}{\Delta}\frac{f^p h}{\varphi^p k}.$$

So that, finally,

$$f(x+h) - f(y+h) \tag{116}$$
$$= \sum_{r=1}^{r=n}\frac{(-1)^{r+1}}{a_r}[\varphi(k + a_r bx) - \varphi(k + a_r by)]\sum_{p=1}^{p=n}\frac{(-1)^{p+1}}{b^p}\frac{\Delta(r,p)}{\Delta}\frac{f^p h}{\varphi^p k} + R.$$

Again, expanding Fx with respect to its first column, we have

$$Fx = f(x+h) - \sum_{p=1}^{p=n}\frac{\Delta_x(p)}{b^p\Delta}\frac{f^p h}{\varphi^p k},$$

whence, since

$$\frac{\Delta_x(p)}{\Delta} = (-1)^{p+1}\sum_{r=1}^{r=p}(-1)^{r+1}\frac{\Delta(r,p)}{a_r\Delta}\varphi(k + a_r bx),$$

in which we change x into y for $\Delta_y(p)/\Delta$, we have

$$f(x+h) - f(y+h) \tag{117}$$
$$= \sum_{p=1}^{p=n}\frac{(-1)^{p+1}}{b^p}\frac{f^p h}{\varphi^p k}\sum_{r=1}^{r=n}\frac{(-1)^{r+1}}{a_r}\frac{\Delta(r,p)}{\Delta}[\varphi(k + a_r bx) - \varphi(k + a_r by)] + R.$$

31. In order to effect the expansion in the case of infinite converging series, it is necessary to evaluate the ratio $\Delta(r,p)/\Delta$ when n is infinite.

Let $\Delta(r,l)$ be Δ in which the rth column and last row have been struck out, then we have

$$\frac{\Delta(r,l)}{\Delta} = \frac{\zeta^{\frac{1}{2}}(a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n)}{\zeta^{\frac{1}{2}}(a_1, \ldots, a_n)},$$

$$= \frac{1}{(a_n - a_r)\ldots(a_{r+1} - a_r)(a_r - a_{r-1})\ldots(a_r - a_1)},$$

$$= \frac{(-1)^{r-1}}{\prod_{(r)\,m=1}^{m=n}(a_m - a_r)};$$

the symbol $\overset{m=n}{\underset{m=1}{\Pi_{(r)}}}$ meaning that, in the binomial product, m is to take all of the integral values from 1 to n, inclusive, except r.

By Jacobi's theorem,* we have

$$\begin{vmatrix} a_1^\alpha, & a_2^\alpha, & \ldots, & a_n^\alpha \\ a_1^\beta, & a_2^\beta, & \ldots, & a_n^\beta \\ \cdot & \cdot & \cdot & \cdot \\ a_1^\xi, & a_2^\xi, & \ldots, & a_n^\xi \end{vmatrix} = \zeta^{\frac{1}{2}}(a_1, \ldots, a_n) \begin{vmatrix} H_\alpha & , H_\beta & , \ldots, & H_\xi \\ H_{\alpha-1} & , H_{\beta-1} & , \ldots, & H_{\xi-1} \\ \cdot & \cdot & \cdot & \cdot \\ H_{\alpha-n+1}, & H_{\beta-n+1}, & \ldots, & H_{\xi-n+1} \end{vmatrix},$$

wherein H_m is the sum of all the products, of weight m, of the quantities $a_1, \ldots, a_n$, with repetition, and $H_0 = 1$, $H_{-m} = 0$. We shall use P_m to represent the sum of all the products of the same quantities, of weight m, without repetition.

We have

$$\frac{\Delta(r,p)}{\Delta} = \frac{(-1)^{r-1}}{\overset{m=n}{\underset{m=1}{\Pi_{(r)}}}(a_m - a_r)} \frac{\Delta(r,p)}{\Delta(r,l)}. \tag{117}$$

In the second member of this equation, divide each column of the determinants $\Delta(r,p)$ and $\Delta(r,l)$ by the constituent of highest degree in that column. Letting these determinants so transformed be represented by $\Delta(r,p)'$ and $\Delta(r,l)'$, we have

$$\Delta(r,l)' = \zeta^{\frac{1}{2}}\left[\frac{1}{a_1}, \ldots, \frac{1}{a_{r-1}}, \frac{1}{a_{r+1}}, \ldots, \frac{1}{a_n}\right];$$

and applying Jacobi's theorem to the determinant $\Delta(r,p)'$, we have

$$\Delta(r,p)' = \Delta(r,l)' \begin{vmatrix} H_1, & H_2, & H_3, & \ldots, & & H_{p-1} \\ 1 & , H_1, & H_2, & \ldots, & & H_{p-2} \\ 0 & ; 1 & , H_1, & \ldots, & & H_{p-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & , 0 & , 0 & , \ldots, & 1, & H_1 \end{vmatrix};$$

wherein, now, H_m is the sum of all the possible powers and products, of degree m, of the quantities

$$\frac{1}{a_1}, \ldots, \frac{1}{a_{r-1}}, \frac{1}{a_{r+1}}, \ldots, \frac{1}{a_n};$$

with a corresponding change in the meaning of the symbol P_m.

* Scott's Determinants, p. 124. Salmon's Higher Algebra (4th Ed.) p. 341.

Therefore (117) becomes

$$\frac{\Delta(r,p)}{\Delta} = \frac{(-1)^{r-1} \overset{m=n}{\underset{m=1}{\Pi_{(r)}}} a_m}{\overset{m=n}{\underset{m=1}{\Pi_{(r)}}} (a_m - a_r)} \begin{vmatrix} H_1, & H_2, & \dots, & H_{p-1} \\ 1, & H_1, & \dots, & H_{p-2} \\ . & . & . & . \\ 0, & 0, & \dots, & H_1 \end{vmatrix}. \tag{118}$$

Let

$$\Xi_{p-1} \equiv \begin{vmatrix} H_1, & H_2, & \dots, & H_{p-1} \\ 1, & H_1, & \dots, & H_{p-2} \\ . & . & . & . \\ 0, & 0, & \dots, & H_1 \end{vmatrix}$$

$$\equiv H_1 \Xi_{p-2} - H_2 \Xi_{p-3} + \dots + (-1)^{m+1} H_m \Xi_{p-m+1} + \dots$$
$$+ (-1)^{p+1} H_{p-2} H_1 + (-1)^p H_{p-1}. \tag{119}$$

We have the known relation*

$$H_{p-1} \equiv \begin{vmatrix} P_1, & P_2, & \dots, & P_{p-1} \\ 1, & P_1, & \dots, & P_{p-2} \\ . & . & . & . \\ 0, & 0, & \dots, & P_1 \end{vmatrix}$$

$$\equiv P_1 H_{p-2} - P_2 H_{p-3} + \dots + (-1)^{m+1} P_m H_{p-m+1} + \dots$$
$$+ (-1)^{p+1} P_{p-2} P_1 + (-1)^p P_{p-1}. \tag{120}$$

It may now be shown, by aid of the recurrence formulæ (119) and (120), through an easy induction, that

$$\Xi_{p-1} = P_{p-1}.$$

Therefore (118) becomes

$$\frac{\Delta(r,p)}{\Delta} = \frac{(-1)^{r-1} \overset{m=n}{\underset{m=1}{\Pi_{(r)}}} a_m}{\overset{m=n}{\underset{m=1}{\Pi_{(r)}}} (a_m - a_r)} P_{p-1}. \tag{121}$$

32. In order that the summation indicated in (115) or (116) may be extended to infinity, or in other words, that the series may be an infinite converging series; we must have the f and φ functions and their derivatives

* See papers by M. Faà de Bruno, Major MacMahon, and Dr. Franklin in the Am. Jour. of Math., Vol. V, No. 3, Vol. VI, No. 3.

finite and existent for all the values of the variable indicated, and also R must become evanescent when n becomes infinitely large. As the first necessary condition for the existence of a converging series, the limit of the ratio $\Delta(r,p)/\Delta$, when n is infinite, must be finite for small values of r and p, diminishing as r and p increase, and ultimately vanishing when r and p become infinitely large. The value of the limit of this ratio is, of course, wholly dependent upon the values which we assign to the arbitraries $a_1, \ldots, a_n$. The determination of this limit is to be effected through the relation (121). The investigation of all the general values of the arbitraries $a_1, \ldots, a_n$ which satisfy the conditions as laid down for the ratio $\Delta(r,p)/\Delta$ is too large a subject to introduce here, and we propose to limit the investigation to one particular set of values of the arbitraries; namely,

$$a_r = r^2 .$$

In this section, therefore, we confine the application of the composite to the consideration of those φ functions which yield the body-determinant

$$\zeta^{\frac{1}{2}}(1^2, 2^2, 3^2, \ldots, n^2) .$$

We have, then, from (121)

$$\frac{\Delta(r,p)}{\Delta} = \frac{(-1)^{r-1} \overset{m=n}{\underset{m=1}{\Pi_{(r)}}} m^2}{\overset{m=n}{\underset{m=1}{\Pi_{(r)}}} (m^2 - r^2)} P_{p-1}$$

$$= \frac{2(n\,!)^2}{(n+r)!\,(n-r)!} P_{p-1}$$

$$= 2P_{p-1} \frac{(1-n^{-1})(1-2n^{-1})\ldots(1-\overline{r-1}\,n^{-1})}{(1+n^{-1})(1+2n^{-1})\ldots(1+rn^{-1})} .$$

When n is infinite, we have

$$\left[\frac{\Delta(r,p)}{\Delta}\right]_{n=\infty} = 2P_{p-1} ,$$

wherein P_{p-1} means the sum of the products, taken $p-1$ at a time without repetition, of the quantities

$$1,\ \frac{1}{2^2},\ \ldots,\ \frac{1}{(r-1)^2},\ \frac{1}{(r+1)^2},\ \ldots,\ \frac{1}{\infty^2},$$

and $P_0 = 1$.

We obtain the value of P_m, at once, by identifying the coefficients of like powers of x, in either pair of the following series:

$$\frac{\sin x}{x} = \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n+1)!} = \prod_1^\infty \left[1 - \frac{x^2}{n^2\pi^2}\right],$$

or

$$\frac{e^x - e^{-x}}{2x} = \sum_0^\infty \frac{x^{2n}}{(2n+1)!} = \prod_1^\infty \left[1 + \frac{x^2}{n^2\pi^2}\right].$$

Whence follows, at once,

$$\left[\frac{\Delta(r,p)}{\Delta}\right]_{n=\infty} = 2\left[\frac{\pi^{2p-2}}{(2p-1)!} - \frac{1}{r^2}\frac{\pi^{2p-4}}{(2p-3)!} + \ldots + (-1)^p \frac{1}{r^{2p-4}}\frac{\pi^2}{3!} + (-1)^{p+1}\frac{1}{r^{2p-2}}\right], \tag{122}$$

$$= \frac{(-1)^{p+1}2}{\pi r^{2p-1}}\left[(\pi r) - \frac{(\pi r)^3}{3!} + \frac{(\pi r)^5}{5!} - \ldots + (-1)^{p+1}\frac{(\pi r)^{2p-1}}{(2p-1)!}\right].$$

Noticing in this last expression that the series of terms within the brackets follows the law of the expansion of the sine, and since r is an integer, we have $\sin(\pi r) = 0$. Therefore we may regard the expression within the brackets as the remainder after p terms of the expansion of the sine of πr with its sign changed. We may therefore write

$$\left[\frac{\Delta(r,p)}{\Delta}\right]_{n=\infty} = \frac{2\pi^{2p}r^2}{(2p+1)!}\sin(\theta\pi r), \qquad (0 < \theta < 1).$$

$$= \frac{2(-1)^{p+1}}{\pi r^{2p-1}}\left[\frac{e^{i\pi r} - e^{-i\pi r}}{2i}\right]_{1 \text{ to } p},$$

where the last parenthesis and its subscript means the first p terms in the expansion of $\sin(\pi r)$.

We notice in this connection that a body-determinant of the form

$$\zeta^{\frac{1}{2}}(1, 2, \ldots, n),$$

fails to comply with the conditions required for converging series, for the ratio of convergency of the body-determinant (if we may so term the limit of the ratio $\Delta(r,p)/\Delta = \rho(r,p)$ when $n = \infty$) becomes, in the case of $a_r = r$,

$$\rho(r,p) = C_{n,r}\, P_{p-1},$$

wherein $C_{n,r}$ is the number of combinations of n things taken r at a time, and P_{p-1} is the sum of the products, taken $p-1$ at a time without repetition, of the quantities

$$1, \frac{1}{2}, \ldots, \frac{1}{r-1}, \frac{1}{r+1}, \ldots, \frac{1}{\infty}.$$

Therefore $\rho(r, p)$ for this case becomes infinite along with n, and the infinite series impossible. We notice also that the form

$$\zeta^{\frac{1}{2}}\left[1\,,\ \frac{1}{2^2}\,,\ \ldots\,,\ \frac{1}{n^2}\right]$$

leads to failure when treated in like manner. On the contrary the body-determinant

$$\zeta^{\frac{1}{2}}\,(2^2\,,\ 4^2\,,\ \ldots\ \overline{2n}^2)$$

leads to the same result as (122).

33. I call the numbers expressed by the formula

$$\sum_{m=0}^{m=n} \frac{(-1)^m}{r^{2m}}\,\frac{\pi^{2n-2-2m}}{(2n-1-2n)\,!}$$

Fourier's numbers after him who first deduced them;* they are destined, I believe, to play a far more important part in analysis. Thus, the above Fourier number is the nth number of the rth order. I shall always use a german $\mathfrak{F}$ to represent Fourier's numbers, indicating the order by a superior prefix, and the number of the order by an inferior suffix. Thus, the above number will be fully represented by the symbol ${}^r\mathfrak{F}_n$.

The numbers of the first, second, and third orders are

$${}^1\mathfrak{F}_1 = 1\,, \quad {}^1\mathfrak{F}_2 = \frac{\pi^2}{3!} - 1\,, \quad {}^1\mathfrak{F}_3 = \frac{\pi^4}{5!} - \frac{\pi^2}{3!} + 1\,, \quad \text{etc.};$$

$${}^2\mathfrak{F}_1 = 1\,, \quad {}^2\mathfrak{F}_2 = \frac{\pi^2}{3!} - \frac{1}{2^2}\,, \quad {}^2\mathfrak{F}_3 = \frac{\pi^4}{5!} - \frac{1}{2^2}\frac{\pi^2}{3!} + \frac{1}{2^4}\,, \quad \text{etc.};$$

$${}^3\mathfrak{F}_1 = 1\,, \quad {}^3\mathfrak{F}_2 = \frac{\pi^2}{3!} - \frac{1}{3^2}\,, \quad {}^3\mathfrak{F}_3 = \frac{\pi^4}{5!} - \frac{1}{3^2}\frac{\pi^2}{3!} + \frac{1}{3^4}\,, \quad \text{etc.}$$

We therefore have

$$\rho(r, p) = \left[\frac{\Delta(r, p)}{\Delta}\right]_{n=\infty} = 2\,{}^r\mathfrak{F}_p\,. \tag{123}$$

That this value for the ratio of convergency of the body-determinant responds to the requirements set forth in § 32 for the existence of convergent series, is easily seen, since it is finite for small values of r and p, diminishes as r and p increase, and ultimately vanishes when r and p become infinite.

* Théorie Analytique de la Chaleur. Paris, 1822. p. 210 et seq.

TABLE OF FOURIER NUMBERS.

${}^r\mathfrak{F}_p$	1	2	3	4	5	6	7	8	9	10
					0.0	0.00	0.000	0.000	0.000	0.000
1	1	0.6449341	0.1668081	0.0249434	012045	11516	9987	— 9916	+ 9919	— 99183
2	1	1.3949341	0.4630087	0.0759993	071481	05691	0106	+ 0047	— 0009	+ 00022
3	1	1.5338230	0.6413174	0.1204950	127596	09384	0486	0017	+ 0001	— 00001
4	1	1.5824341	0.7128401	0.1471990	169480	12969	0719	0026	0001	000001
5	1	1.6049341	0.7475448	0.1618497	196739	15691	0901	0035	0001	000002
6	1	1.6171563	0.7668212	0.1704509	214131	17612	1040	0042	0001	000003
7	1	1.6245259	0.7785886	0.1758618	225589	18957	1142	0048	0002	000004
8	1	1.6293091	0.7862843	0.1794789	233435	19914	1218	0052	0002	000005
9	1	1.6325874	0.7915868	0.1819788	239013	20610	1275	0055	0002	000005
10	1	1.6349341	0.7953929	0.1837976	243099	21130	1318	0058	0002	000005
∞	1	1.6449341	0.8117422	0.1917515	261479	23561	1529	0071	0003	000007

The table is easily computed from the formula

$$ {}^r\mathfrak{F}_p + \frac{1}{r^2}\,{}^r\mathfrak{F}_{p-1} = {}^\infty\mathfrak{F}_p = \frac{\pi^{2p-2}}{(2p-1)!}\,. $$

The rows correspond to the values of p and the columns to those of r. For p fixed ${}^r\mathfrak{F}_p$ soon approaches the constant limit $\pi^{2p-2}/(2p-1)!$; and for r finite and fixed ${}^r\mathfrak{F}_p$ soon approaches the value ${}^r\mathfrak{F}_p = -\,{}^r\mathfrak{F}_{p-1}/r^2$, which is alternately positive and negative; these values are just beginning to appear in the upper right hand corner of the table.

We have then for (116), finally,

$$ f(x+h) = f(y+h) + \sum_{r=1}^{r=n} (-1)^{r+1} \frac{A_r}{a_r} \left[\varphi_r(k + a_r bx) - \varphi(k + a_r by)\right] + R\,, $$

$$ A_r = \sum_{p=1}^{p=n} \frac{B_p}{b_p} \frac{f^p h}{\varphi^p k}\,, $$

$$ (-1)^{p+1} B_p = \frac{(-1)^{r-1} \Pi_m a_m}{\Pi_m (a_m - a_r)} P_{p-1}\,, \qquad (m = 1, \ldots, r-1, r+1, \ldots, n)\,, $$

and P_{p-1} is the sum of the products, $p-1$ at a time without repetition, of the quantities

$$a_1^{-1}, \quad \ldots, \quad a_{n-1}^{-1}, \quad a_{n+1}^{-1}, \ldots, \quad a_n^{-1}.$$

When $n=\infty$ and $a_r = r^2$, we have

$$(+1)^{p+1} B_p = 2\,{}^r\mathfrak{F}_p .$$

We notice, in passing, that if we put $a_r = a^r$, we find

$$(-1)^{p+1} B_p = \frac{P_{p-1}^{\bullet}}{(a-1)\ldots(a^r-1)(1-a^{-1})\ldots(1-a^{-(n-r)})},$$

in which, if we have $a^2 > 1$ and $n=\infty$, $(1-a^{-1})\ldots(1-a^{-(n-r)})$ has a finite limit, and

$$P_0 = 1,$$

$$P_1 = \frac{1}{a-1} - \frac{1}{a^r}$$

$$P_2 = \frac{1}{(a-1)(a^2-1)} - \frac{1}{a^r}\left[\frac{1}{a-1} - \frac{1}{a^r}\right],$$

and in general,

$$P_r = \frac{1}{(a-1)\ldots(a^r-1)} - \frac{1}{a^r} P_{r-1};$$

since it is easy to show that the sum of the products, r at a time without repetition, of the quantities $a^{-1}, a^{-2}, \ldots, a^{-\infty}$ is $[(a-1)\ldots(a^r-1)]^{-1}$. These interesting series we take up again further on. It is to be remembered that, if the functions φ be even functions, and we put $h=0$, $k=0$, we are to throw out the odd derivative rows in the composite. If odd functions, we cast out the even derivative rows, in virtue of § 4.

34. We shall have occasion to need the following three summations in connection with Fourier's numbers :—

$$\sum_{r=1}^{r=\infty} \frac{(-1)^{r+1}}{r^2}\,{}^r\mathfrak{F}_n = \sum_{r=1}^{r=\infty}\sum_{m=0}^{m=n} \frac{(-1)^{r+1}(-1)^m}{r^{2m+2}} \frac{\pi^{2n-2-2m}}{(2n-1-2m)!}$$

$$= \pi^{2n} \sum_{m=1}^{m=n} (-1)^{m+1} \frac{2^{2m-1}-1}{(2n-2m+1)!} \frac{B_m}{(2m)!}$$

$$= \frac{\pi^{2n}}{2(2n+1)!}, \qquad (124)$$

$$\sum_{r=1}^{r=\infty} \frac{1}{(2r-1)^2}\, {}^{2r-1}\mathfrak{F}_n = \sum_{r=1}^{r=\infty} \sum_{m=1}^{m=n} \frac{(-1)^{m+1}}{(2r-1)^{2m}} \frac{\pi^{2n-2m}}{(2n-2m+1)!}$$

$$= \frac{\pi^{2n}}{2} \sum_{m=1}^{m=n} (-1)^{m+1} \frac{2^{2m}-1}{(2n-2m+1)!} \frac{B_m}{(2m)!}$$

$$= \frac{\pi^{2n}}{4(2n)!}, \tag{125}$$

$$\sum_{r=1}^{r=\infty} \frac{1}{r^2}\, {}^{r}\mathfrak{F}_n = \sum_{r=1}^{r=\infty} \sum_{m=1}^{m=n} \frac{(-1)^{m+1}}{r^2} \frac{\pi^{2n-2m}}{(2n-2m+1)!}$$

$$= \pi^{2n} \sum_{m=1}^{m=n} (-1)^{m+1} \frac{2^{2m-1}}{(2n-2m+1)!} \frac{B_m}{(2m)!}$$

$$= \frac{n}{(2n+1)!} \pi^{2n}, \tag{126}$$

in which we have used these three known formulæ in Bernoulli's numbers; i. e.,

$$\sum_{r=1}^{r=\infty} \frac{1}{r^{2m}} = \frac{2^{2m-1}\pi^{2m}}{(2m)!} B_m, \qquad \sum_{r=1}^{r=\infty} \frac{1}{(2r-1)^{2m}} = \frac{2^{2m}-1}{2(2m)!} B_m \pi^{2m},$$

$$\sum_{r=1}^{r=\infty} \frac{(-1)^{r+1}}{r^{2m}} = \frac{2^{2m-1}-1}{(2m)!} B_m \pi^{2m}.$$

To prove (124), (125) and (126) we lay down the three known series

$$\frac{x}{e^x+1} = \frac{1}{2}x - (2^2-1)\frac{x^2}{2!}B_1 + (2^4-1)\frac{x^4}{4!}B_2 + \dots, \tag{i}$$

$$\frac{x}{e^x-1} = 1 - \frac{1}{2}x + \frac{x^2}{2!}B_1 - \frac{x^4}{4!}B_2 + \dots, \tag{ii}$$

$$\frac{e^{2x}-1}{2xe^x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \dots. \tag{iii}$$

Add (i) and (ii), and multiply the result by (iii); whence

$$0 \equiv \left[-\frac{2^1-1}{2!}B_1 + \frac{1}{2}\frac{1}{3!} \right] x^2 + \dots$$

$$+ \left[\sum_{m=1}^{m=n} (-1)^m \frac{2^{2m-1}-1}{2n-2m+1)!} \frac{B_m}{(2m)!} + \frac{1}{2(2n+1)!} \right] x^{2n} + \dots;$$

$$\therefore \quad \sum_{m=1}^{m=n} (-1)^{m+1} \frac{2^{2m-1}-1}{(2n-2m+1)!} \frac{B_m}{(2m)!} = \frac{1}{2(2n+1)!}. \tag{a}$$

Multiply together (i) and (iii), and from the result subtract

$$\frac{1}{2}(1-e^{-x})=\frac{1}{2}x-\frac{1}{2}\frac{x^2}{2!}+\frac{1}{2}\frac{x^3}{3!}-\ldots;$$

whence

$$0=\left[\frac{1}{2}\frac{1}{2!}-\frac{2^2-1}{1!}\frac{B_1}{2!}\right]x^2+\ldots$$

$$+\left[\frac{1}{2}\frac{1}{(2n)!}-\sum_{m=1}^{m=n}(-1)^{m+1}\frac{2^{2m}-1}{(2n-2m+1)!}\frac{B_m}{(2m)!}\right]x^{2n}+\ldots;$$

$$\therefore\quad \sum_{m=1}^{m=n}(-1)^{m+1}\frac{2^{2m}-1}{(2n-2m+1)!}\frac{B_m}{(2m)!}=\frac{1}{2(2n)!}. \tag{b}$$

Put $2x$ for x in (ii), and multiply the result by (iii); from the resulting equation subtract

$$e^{-x}=1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\ldots;$$

whence

$$0=\left[\frac{1}{3!}-\frac{1}{2!}+\frac{2^2}{1!}\frac{B_1}{2!}\right]x^2+\ldots$$

$$+\left[\sum_{m=1}^{m=n}(-1)^{m+1}\frac{2^{2m}}{(2n-2m+1)!}\frac{B_m}{(2m)!}-\frac{2n}{(2n+1)!}\right]x^{2n}+\ldots;$$

$$\therefore\quad \sum_{m=1}^{m=n}(-1)^{m+1}\frac{2^{2m-1}}{(2n-2m+1)!}\frac{B_m}{(2m)!}=\frac{n}{(2n+1)!}. \tag{c}$$

The results (a), (b), and (c) establish the formulæ.

35. We may now write down the expansion formula

$$f(x+h)-f(y+h)$$
$$=2\sum_{r=1}^{r=\infty}\frac{(-1)^{r+1}}{r^2}[\varphi(k+r^2bx)-\varphi(k+r^2by)]\sum_{p=1}^{p=\infty}\frac{(-1)^{p+1}}{b^p}\frac{f^ph}{\varphi^pk}\,{}_r\mathfrak{F}_p, \tag{127}$$

under the assumption that R has become evanescent with the infinite value of n. While aware of the incompleteness of the formula until this assumption has been thoroughly investigated, we defer its consideration for the present. Particular values assigned to the arbitrary constants h, k, and b, yield special formulæ.

We now proceed to give examples of the forms of expansions according to the general theorem exposed in the preceding articles; but, in order to better illustrate the method of operation, these particular series will usually be

deduced directly from the composite itself, rather than through substitution in a less general formula.

Sines and Cosines.

36. The subject we are about to take up is one of the most remarkable applications of the Integral Calculus, as it is usually presented, and its treatment may be taken to be a type of the first method of obtaining series, referred to in § 27. A handy reference to its elementary treatment is Todhunter's Integral Calculus, p. 295.

Consider the expansion of fx in terms of the functions cos rax, or rather

$$\frac{1}{r^2} \cos rax \,,$$

(for, as the sequel shows, we factor out the r^2, so that the functions really take the second form), r taking in succession the values 1, 2, 3, . . . , and a being any arbitrary constant.

Since these functions are even functions, containing no odd powers of the variable, the composite will contain no odd derivative forms, in virtue of the theorem of § 4.

We have, therefore, after substituting the functions in the composite,

$$\begin{vmatrix} fx, & 1, & \cos ax, & \frac{1}{2^2}\cos 2ax, & \frac{1}{3^2}\cos 3ax, & \ldots, & \frac{1}{n^2}\cos nax, & \frac{1}{(n+1)^2}\cos(n+1)ax \\ fy, & 1, & \cos ay, & \frac{1}{2^2}\cos 2ay, & \frac{1}{3^2}\cos 3ay, & \ldots, & \frac{1}{n^2}\cos nay, & \frac{1}{(n+1)^2}\cos(n+1)ay \\ f''x_2, & 0, & -a^2\cos ax_2, & -a^2\cos 2ax_2, & -a^2\cos 3ax_2, & \ldots, & -a^2\cos nax_2, & -a^2\cos(n+1)ax_2 \\ f^{\text{iv}}x_4, & 0, & +a^4\cos ax_4, & +a^4 2^2\cos 2ax_4, & +a^4 3^2\cos 3ax_4, & \ldots, & +a^4 n^2\cos nax_4, & +a^4(n+1)^2\cos(n+1)ax_4 \\ f^{\text{vi}}x_6, & 0, & -a^6\cos ax_6, & -a^6 2^4\cos 2ax_6, & -a^6 3^4\cos 3ax_6, & \ldots, & -a^6 n^4\cos nax_6, & -a^6(n+1)^2\cos(n+1)ax_6 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f^{2n}x_{2n}, & 0, & \pm a^{2n}\cos ax_{2n}, & \pm a^{2n}2^{2n-2}\cos 2ax_{2n}, & \pm a^{2n}3^{2n-2}\cos 3ax_{2n}, & \ldots, & \pm a^{2n}n^{2n-2}\cos nax_{2n}, & \pm a^{2n}(n+1)^{2n-2}\cos(n+1)ax_{2n} \\ \Phi(u), & 0, & 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0. \; ($$

Let $x_2 = x_4 = \ldots x_{2n} = z$; then, factoring out the common factors from the body-determinant, we notice that we may substitute the functions $\cos rax / r^2 \cos raz$ in the composite at once. We have, thus,

$$\begin{vmatrix} fx & , 1 , & \frac{\cos ax}{\cos az}, & \frac{1}{2^2}\frac{\cos 2ax}{\cos 2az}, & \ldots, & \frac{1}{n^2}\frac{\cos nax}{\cos naz}, & \frac{1}{(n+1)^2}\frac{\cos (n+1)ax}{\cos (n+1)az} \\ fy & , 1 , & \frac{\cos ay}{\cos az}, & \frac{1}{2^2}\frac{\cos 2ay}{\cos 2az}, & \ldots, & \frac{1}{n^2}\frac{\cos nay}{\cos naz}, & \frac{1}{(n+1)^2}\frac{\cos (n+1)ay}{\cos (n+1)az} \\ \frac{1}{a^2}f''z & , 0 , & 1 , & 1 & , \ldots, & 1 , & 1 \\ \frac{1}{a^4}f^{\mathrm{iv}}z & , 0 , & 1 , & 2^2 & , \ldots, & n^2 , & (n+1)^2 \\ \frac{1}{a^6}f^{\mathrm{vi}}z & , 0 , & 1 , & 2^4 & , \ldots, & n^4 , & (n+1)^4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{a^{2n}}f^{2n}z & , 0 , & 1 , & 2^{2n-2} & , \ldots, & n^{2n-2} , & (n+1)^{2n-2} \\ \Phi(u) & , 0 , & 0 , & 0 & , \ldots, & 0 , & 1 \end{vmatrix} = 0 . \quad (129)$$

Expanding this with respect to the first and second rows, we obtain

$$fx - fy = \sum_1^n \frac{(-1)^{r+1}A_r}{r^2 \cos raz}(\cos rax - \cos ray) + R , \qquad (130)$$

in which A_r does not contain x.

This may be written

$$fx - fy = 2 \sum_1^n \frac{(-1)^r A_r}{r^2 \cos raz} \sin \tfrac{1}{2} ra(x+y) \sin \tfrac{1}{2} ra(x-y) + R . \qquad (131)$$

In this put $x+h$ for x, and x for y, whence

$$f(x+h) = fx + 2 \sum_1^n \frac{(-1)^r A_r}{r^2 \cos raz} \sin \tfrac{1}{2} rah \sin ra\,(x + \tfrac{1}{2}h) + R . \qquad (132)$$

In this put $x = 0$ and $h = x$, whence

$$fx = f0 + 2 \sum_1^n \frac{(-1)^r A_r}{r^2 \cos raz} \sin^2 \tfrac{1}{2}(rax) + R . \qquad (133)$$

The value of A_r in these four formulæ is, when n is infinite,

$$A_r = -2 \sum_{p=1}^{p=\infty} \frac{1}{a^{2p}}\, {}^r\mathfrak{F}_p f^{2p}z .$$

In (130) put $z = y = l$, and $a = 2\pi/l$; then, since $\cos ral = 1$, we have

$$fx = A_0 + 2 \sum_{r=1}^{r=n} \frac{(-1)^r}{r^2} A_r \cos \frac{2\pi r}{l} x + R , \qquad (134)$$

wherein, when n is infinite, we have for the constant term

$$A_0 = f(l) + 2 \sum_{r=1}^{r=\infty} \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}}{r^2} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p}(l)$$

$$= f(l) + \sum_{p=1}^{p=\infty} \frac{1}{(2p+1)!} \left[\frac{l}{2}\right]^{2p} f^{2p}(l)$$

$$= \frac{1}{l} \int_{\frac{1}{2}l}^{\frac{3}{2}l} fx dx ,$$

in virtue of (124) and series (45).

The value of A_r is

$$A = \sum_{p=1}^{p=\infty} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p} z .$$

In like manner, if we had put $z = y = 0$, the constant term would have become

$$A'_0 = \sum_0^\infty \frac{1}{(2p+1)!} \left[\frac{l}{2}\right]^{2p} f^{2p}(0) = \frac{1}{l} \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} fx dx ;$$

and the substitution $z = y = \frac{1}{2} l$ gives

$$A_0'' = \sum_0^\infty \frac{1}{(2p+1)!} \left[\frac{l}{2}\right]^{2p} f^{2p}(0) = \frac{1}{l} \int_0^l fx dx ,$$

by series (47).

If, instead of $2\pi/l$, we put $a = \pi/l$; then, since $\cos r\pi = (-1)^r$, we have

$$fx = B_0 - 2 \sum_{r=1}^{r=n} \frac{1}{r^2} B_r \cos \frac{\pi r}{l} x + R , \tag{135}$$

wherein the constant term is, for $n = \infty$, $z = l$, $z = y = 0$, and $z = y = \frac{1}{2} l$, respectively,

$$B_0 = \sum_0^\infty \frac{1}{(2p+1)!} l^{2p} f^{2p}(l) = \frac{1}{2l} \int_0^{2l} fx dx ,$$

$$B_0' = \sum_0^\infty \frac{1}{(2p+1)!} l^{2p} f^{2p}(0) = \frac{1}{2l} \int_{-l}^{+l} fx dx ,$$

$$B_0'' = \sum_0^\infty \frac{1}{(2p+1)!} l^{2p} f^{2p}(\tfrac{1}{2} l) = \frac{1}{2l} \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} fx dx ;$$

the effect of the different substitutions being to change the limits between which the function is to be considered.

Expressions (134) and (135) are the equivalents of the familiar forms of the expansion in cosines.

If, in (134), we put $z = y = \frac{1}{2} l$, and multiply through by $2 \cos (2\pi r x/l)$, and integrate both sides from $x = 0$ to $x = l$, we get

$$\frac{1}{l} \int_0^l fx \cos \frac{2\pi r}{l} x \, dx = 2 \sum_{p=1}^{p=\infty}{}' \frac{1}{r^2} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi} \right]^{2p} f^{2p}(\tfrac{1}{2} l) ,$$

the definite integral being the form of the coefficient as usually given.

It is to be observed that, if we make the substitution $z = y = \frac{1}{2} l$ and $a = 2\pi/l$ in (128), the signs in (134) become the same as the signs in (135), since $\cos (r\pi) = (-1)^r$; while the substitution $z = y = \frac{1}{2} l$ and $a = \pi/l$ requires us to throw out all columns containing odd values of r, since for these values $\cos \frac{1}{2} r\pi$ vanishes.

37. Consider the expansion of fx in terms of the functions $\sin rax$, or

$$\frac{1}{r} \sin rax ,$$

since the columns will factor to this shape.

Remembering that in this case the composite can contain no even derivative rows in virtue of § 4, and proceeding exactly as in the last article, we deduce the formula

$$fx - fy = \sum_1^n{}' \frac{(-1)^{r+1} A_r}{r \cos raz} (\sin rax - \sin ray) + R \tag{136}$$

$$= 2 \sum_1^n{}' \frac{(-1)^{r+1} A_r}{r \cos raz} \cos \tfrac{1}{2} ra(x+y) \sin \tfrac{1}{2} ra(x-y) + R, \tag{137}$$

in which when n is infinite, we have

$$A_r = 2 \sum_{p=1}^{p=\infty}{}' \frac{1}{a^{2p-1}} {}^r\mathfrak{F}_p f^{2p-1}(z).$$

Put $a = 2\pi/l$ and $z = y = l$; then, since $\cos 2\pi r = 1$, we have

$$fx = f(l) + 2 \sum_{r=1}^{r=\infty}{}' \frac{(-1)^{r+1}}{r} \sin \frac{2\pi r}{l} x \sum_{p=1}^{p=\infty}{}' {}^r\mathfrak{F}_p \left[\frac{l}{2\pi} \right]^{2p-1} f^{2p-1}(l) , \tag{138}$$

assuming that n may be infinite.

If we put $z = y = \frac{1}{2} l$ instead of l, we have, since $\cos r\pi = (-1)^r$,

$$fx = f(l) + 2 \sum_{r=1}^{r=\infty}{}' - \sin \frac{2\pi r}{l} x \sum_{p=1}^{p=\infty}{}' \frac{1}{r} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi} \right]^{2p-1} f^{2p-1}(\tfrac{1}{2}l) . \tag{139}$$

If fx be a function which vanishes with x, then $f(l) = 0$, and we may write (139)

$$f(-x) = \sum_{r=1}^{r=\infty} A_r \sin \frac{2\pi r}{l} x \,, \tag{140}$$

wherein

$$A_r = 2 \sum_{p=1}^{p=\infty} \frac{1}{r} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi} \right]^{2p-1} f^{2p-1}(\tfrac{1}{2}l) \,.$$

Multiply both sides of (140) by $2 \sin (2\pi r x/l)$ and integrate from $x = 0$ to $x = l$, whence

$$\frac{1}{l} \int_0^l fx \sin \frac{2\pi r}{l} x = -2 \sum_{p=1}^{p=\infty} \frac{1}{r} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi} \right]^{2p-1} f^{2p-1}(\tfrac{1}{2}l) \,,$$

the definite integral being the usual form of this coefficient.

We may derive Fourier's expansion of fx as a complex harmonic function, at once, by the addition of (140) and (134), but we deduce it directly from the composite on account of its importance and historical connection with this method of treatment.

38. Consider the expansion of fx in terms of the functions

$$\frac{1}{r^2} \cos rax \quad \text{and} \quad \frac{1}{r} \sin rax \,.$$

Making the substitutions in the composite, we have,

$$\left| \begin{array}{llllll}
fx & , 1 , \cos ax & , \sin ax & , \ldots , \cos nax & , \sin nax & , (\mathrm{c}, \mathrm{s}) \\
fy & , 1 , \cos ay & , \sin ay & , \ldots , \cos nay & , \sin nay & , (\mathrm{c}, \mathrm{s}) \\
f'x_1 & , 0 , -a \sin ax_1 & , +a \cos ax_1 & , \ldots , -na \sin nax_1 & , +na \cos nax_1 & , (n+1)a\, (\mathrm{s}, \mathrm{c}) \\
f''x_2 & , 0 , -a^2 \cos ax_2 & , -a^2 \sin ax_2 & , \ldots , -n^2a^2 \cos nax_2 & , -n^2a^2 \sin nax_2 & , (n+1)^2a^2\, (\mathrm{c}, \mathrm{s}) \\
f'''x_3 & , 0 , +a^3 \sin ax_3 & , -a^3 \cos ax_3 & , \ldots , +n^3a^3 \sin nax_3 & , -n^3a^3 \cos nax_3 & , (n+1)^3a^3\, (\mathrm{s}, \mathrm{c}) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
f^{2n-1}x_{2n-1} & , 0 , \mp a^{2n-1} \sin ax_{2n-1} & , \pm a^{2n-1} \cos ax_{2n-1} & , \ldots , \mp (na)^{2n-1} \sin nax_{2n-1} & , \pm (na)^{2n-1} \cos nax_{2n-1} & , (\overline{n+1}\,.\,a)^{2n-1} (\mathrm{s}, \mathrm{c}) \\
f^{2n}x_{2n} & , 0 , \mp a^{2n} \cos ax_{2n} & , \mp a^{2n} \sin ax_{2n} & , \ldots , \mp (na)^{2n} \cos nax_{2n} & , \mp (na)^{2n} \sin nax_{2n} & , (\overline{n+1}\,.\,a)^{2n} (\mathrm{c}, \mathrm{s}) \\
\Phi(u) & , 0 , \quad 0 & , \quad 0 & , \ldots , \quad 0 & , \quad 0 & , \quad 1
\end{array} \right| = 0. \tag{141}$$

The symbol (c, s) means that either cosine or sine may be used in the last column. In (141) put $a = 2\pi/l$, and

$$x_1 = x_2 = \ldots = x_{2n-1} = x_{2n} = \tfrac{1}{2}\, l$$

(noticing as we make this substitution what would result if, instead, we made the substitution $x_1 = \ldots = x_{2n} = 0$, or l).

We have throughout the body-determinant, $\sin r\pi = 0$ and $\cos r\pi = (-1)^r$. Factor the a's from the rows, and the negative signs out of the body-determinant; continuing to write a for $2\pi/l$ for brevity. Expand (141) according to the method of § 30, and consider the determinant Fx. In Fx, run the first column and row to the middle, the sines to the left and cosines to the right, the odd derivative forms to the top and the even ones to the bottom; thus obtaining Fx in the following shape:

$$\left|\begin{array}{cccccccccc}
1, & 1, & \ldots, & 1, & -\frac{1}{a}f'(\tfrac{1}{2}l), & 0, & 0, & \ldots, & 0 \\
1^2, & 2^2, & \ldots, & n^2, & +\frac{1}{a^3}f'''(\tfrac{1}{2}l), & 0, & 0, & \ldots, & 0 \\
1^4, & 2^4, & \ldots, & n^4, & -\frac{1}{a^5}f^{\mathrm{v}}(\tfrac{1}{2}l), & 0, & 0, & \ldots, & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1^{2n-2}, & 2^{2n-2}, & \ldots, & n^{2n-2}, & \pm\frac{1}{a^{2n-1}}f^{2n-1}(\tfrac{1}{2}l), & 0, & 0, & \ldots, & 0 \\
\sin ax, & -\frac{1}{2}\sin 2ax, & \ldots, & \pm\frac{1}{n}\sin nax, & fx, & \cos ax, & -\frac{1}{2^2}\cos 2ax, & \ldots, & \pm\frac{1}{n^2}\cos nax \\
0, & 0, & \ldots, & 0, & +\frac{1}{a^2}f''(\tfrac{1}{2}l), & 1, & 1, & \ldots, & 1 \\
0, & 0, & \ldots, & 0, & -\frac{1}{a^4}f^{\mathrm{iv}}(\tfrac{1}{2}l), & 1^2, & 2^2, & \ldots, & n^2 \\
0, & 0, & \ldots, & 0, & +\frac{1}{a^6}f^{\mathrm{vi}}(\tfrac{1}{2}l), & 1^4, & 2^4, & \ldots, & n^4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0, & 0, & \ldots, & 0, & \pm\frac{1}{a^{2n}}f^{2n}(\tfrac{1}{2}l), & 1^{2n-2}, & 2^{2n-2}, & \ldots, & n^{2n-2}
\end{array}\right| . \qquad (142)$$

Expanding this with respect to the middle row, we have

$$fx - \sum_{r=1}^{r=n}\left\{A_r \cos\frac{2\pi r}{l}x + B_r \sin\frac{2\pi r}{l}x\right\}.$$

We therefore derive the formula

$$fx - fy = \sum_{r=1}^{r=n}\left\{A_r\left[\cos\frac{2\pi r}{l}x - \cos\frac{2\pi r}{l}y\right]\right.$$

$$\left. + B_r\left[\sin\frac{2\pi r}{l}x - \sin\frac{2\pi r}{l}y\right]\right\} + R, \qquad (143)$$

wherein, when n is infinite,

$$A_r = 2 \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}}{r^2} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p}(\tfrac{1}{2}l),$$

$$B_r = 2 \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}}{r} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p-1} f^{2p-1}(\tfrac{1}{2}l).$$

Put $y = \frac{1}{2} l$, then $\cos \pi r = (-1)^r$, and we have for the constant term

$$A_0 = \sum_0^\infty \frac{1}{(2p+1)!} \left[\frac{l}{2}\right]^{2p} f^{2p}(\tfrac{1}{2}l) = \frac{1}{l} \int_0^l fx dx ;$$

$$\therefore \qquad fx = A_0 + \sum_1^\infty \left[A_r \cos \frac{2\pi r}{l} x + B_r \sin \frac{2\pi r}{l} x \right], \qquad (144)$$

wherein A_0, A_r, B_r have the values assigned above, and (144) is the standard form of Fourier's Theorem.

EXPONENTIAL FORMS.

39. We frequently meet such expressions as "If fx be a function developable in powers of e^x;" for example, the ordinary deduction of Abel's series is dependent upon this assumption; i. e.,

$$fx = A_0 + A_1 e^x + A_2 e^{2x} + \ldots + A_r e^{rx} + \ldots,$$

See Carr's Synopsis, p. 282. Again, in the same work, p. 394, we read, "Given that $F(x+a)$ can be expanded in powers of e^{-a}, then" Abel's formula for the definite integral of a complex function follows, based upon the assumption

$$F(x+a) = A_0 + A_1 e^{-a} + \ldots + A_r e^{-ra} + \ldots.$$

Our method shows clearly that these two expansions are impossible when we attempt them, by substitution in the *complete* composite, because when we attempt the expansion of fx according to the functions $e^{\pm rx}$ it is at once seen that these functions yield the body-determinant

$$\zeta^{\frac{1}{2}}(1, 2, \ldots, n),$$

for which, the value of the ratio $\rho(r, p)$ has been shown, § 32, to become infinite with n. We return to this form in § 42.

In Todhunter's Functions, p. 128, in an example determining the temperature of a homogeneous sphere placed in a medium of constant temperature,

we read, " We will also assume that as u is a function of t, it may be expanded in a series proceeding according to ascending powers of e^{-t}; this assumption may in some degree be justified by Burmann's Theorem. We assume, then, that u can be expressed in a series of the form

$$u = A_1 e^{-a_1^2 t} + A_2 e^{-a_2^2 t} + \ldots + A_r e^{-a_r^2 t} + \ldots ."$$

This expression, it is easy to see, is a possible one when the arbitraries have the values given by $a_r = r$; for then the functions $e^{-r^2 t}$ yield, in the composite, a body-determinant

$$\zeta^{\frac{1}{2}}(1^2, 2^2, \ldots, n^2) .$$

We may, therefore, express the coefficients A_r in a converging infinite series in terms of Fourier's numbers and derivative forms of u.

40. Consider the expansion of fx in terms of the functions $e^{-r^2 ax}$, in which a is an arbitrary constant, and r takes successively the values 1, 2, 3, ... Making the substitutions in the composite and putting all of the arbitraries equal to z after differentiation, factoring all common factors and negative signs out of the body-determinant, we obtain

$$\left|\begin{array}{lllllll} fx & , 1 , & e^{a(z-x)}, & \frac{1}{2^2} e^{2^2 a(z-x)}, & \ldots, & \frac{1}{n^2} e^{n^2 a(z-x)}, & \frac{1}{(n+1)^2} e^{(n+1)^2 a(z-x)} \\ fy & , 1 , & e^{a(z-y)}, & \frac{1}{2_2} e^{2^2 a(z-y)}, & \ldots, & \frac{1}{n^2} e^{n^2 a(z-y)}, & \frac{1}{(n+1)^2} e^{(n+1)^2 (z-y)} \\ -\frac{1}{a} f'z & , 0 , & 1 , & 1 & , \ldots, & 1 & , \quad 1 \\ +\frac{1}{a^2} f''z & , 0 , & 1 , & 2^2 & , \ldots, & n^2 & , \quad (n+1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{(-1)^n}{a^n} f^n z & , 0 , & 1 , & 2^{2n-2} & , \ldots, & n^{2n-2} & , \quad (n+1)^{2n-2} \\ \Phi(u) & , 0 , & 0 , & 0 & , \ldots, & 0 & , \quad 1 \end{array}\right| = 0. \tag{145}$$

Expanding this, we obtain the formula

$$fx - fy = 2 \sum_{r=1}^{r=n}{}' (e^{r^2 a(z-x)} - e^{r^2 a(z-y)}) A_r + R , \tag{146}$$

wherein, for $n = \infty$,

$$A_r = \sum_{p=1}^{p=\infty}{}' \frac{(-1)^r}{r^2 a^p} {}^r\mathfrak{F}_p \, f^p z ,$$

a true series when the exponents of e are negative.

If $z = y = l$, we obtain

$$fx = A_0 + \sum_{r=1}^{r=n}{}' A_r e^{r^2 a(l-x)} + R, \tag{147}$$

wherein the values of the constants are obvious.

In (147) put* $a = ib$, where i is the operator $\sqrt{-1}$; and also put

$$e^{r^2 ib(l-x)} = \cos r^2 b(l-x) + i \sin r^2 b(l-x);$$

whence

$$fx = (A + iB) + 2 \sum_{r=1}^{r=n}{}' [\cos r^2 b(l-x) - i \sin r^2 b(l-x)](A_r + iB_r) + R,$$

in which, when $n = \infty$,

$$A_0 = f(l) + 2 \sum_{r=1}^{r=\infty}{}' \sum_{p=1}^{p=\infty}{}' \frac{(-1)^{r+1}(-1)^{p+1}}{r^2 b^{2p}} {}^r\mathfrak{F}_{2p} f^{2p}(l),$$

$$B_0 = 2 \sum_{r=1}^{r=\infty}{}' \sum_{p=1}^{p=\infty}{}' \frac{(-1)^{r+1}(-1)^{p+1}}{r^2 b^{2p-1}} {}^r\mathfrak{F}_{2p-1} f^{2p-1}(l),$$

$$A_r = \sum_{p=1}^{p=\infty}{}' \frac{(-1)^r(-1)^{p+1}}{r^2 b^{2p}} {}^r\mathfrak{F}_{2p} f^{2p}(l),$$

$$B_r = \sum_{p=1}^{p=\infty}{}' \frac{(-1)^r(-1)^{p+1}}{r^2 b^{2p-1}} {}^r\mathfrak{F}_{2p-1} f^{2p-1}(l).$$

By equating real and imaginary parts, if fx be a real function, then follow the series

$$fx = A_0 + 2 \sum_{r=1}^{r=n}{}' [A_r \cos r^2 b(l-x) + B_r \sin r^2 b(l-x)] + R, \tag{148}$$

$$0 = B_0 + 2 \sum_{r=1}^{r=n}{}' [B_r \cos r^2 b(l-x) - A_r \sin r^2 b(l-x)] + R.$$

Simpler forms follow when $b = 2\pi r/l$ or $b = 2\pi r/m$ and $l = 0$.

40. The function e^{-x^2} is a very important one in mathematical physics, it becomes unity when x vanishes, and vanishes itself an infinite number of times when $x = \pm \infty$.

The function

$$\varphi(x) = \frac{d^n e^{-x^2}}{dx^n} = H_n e^{-x^2},$$

* This substitution, introducing unreal quantities, is here permissible only as a matter of form. As yet we have not demonstrated the composite relation wherein complex quantities are involved. This demonstration will be given in another place for analytical functions of a complex variable; it differs but slightly from that for functions of a real variable.

is equal to the product of e^{-x^2} into a rational integral function of x of the nth degree, which we have indicated by H_n. These functions have been studied by M. Hermite, and according to custom we call them Hermite's functions (Laurent, Traité d'Analyse, T. V. p. 213).

Consider the expansion of fx in terms of the functions $e^{-ar^2x^2}$, a being an arbitrary positive constant, and r taking successively the values 1, 2, 3, ...

Put

$$y = e^{-ar^2x^2};$$

$$\therefore \qquad \log y = -ar^2x^2, \qquad \text{and} \qquad \frac{dy}{dx} = -2ar^2xy.$$

Differentiating $n + 1$ times, we obtain

$$\frac{d^{n+2}y}{dx^{n+2}} + 2ar^2x\frac{d^{n+1}y}{dx^{n+1}} + 2ar^2(n+1)\frac{d^ny}{dx^n} = 0.$$

Dividing through by $e^{-ar^2x^2}$, we obtain the following relation between three consecutive functions of Hermite:

$$H_{n+2} + 2ar_2x\,H_{n+1} + 2ar^2(n+1)\,H_n = 0;$$

and if $x = 0$, we have

$${}^0H_{n+2} = -2ar^2(n+1)\,{}^0H_n.$$

Since ${}^0H_1 = 0$, we have ${}^0H_{2n-1} = 0$, and since ${}^0H_2 = -2ar^2$ it is easy to see that

$${}^0H_{2n} = (-1)^{n+1}\,2^na^nr^{2n}.\,3\,.\,5\,.\,7\,\ldots\,(2n-1)$$

$$= (-1)^{n+1}\,a^nr^{2n}\frac{(2n)!}{n!}.$$

Since on factoring the common factors out of the body-determinant we divide r^2 out of the rth column, we may regard the expansion of fx as effected according to the functions $r^{-2}\,e^{-ar^2x^2}$.

Making the proper substitutions in the composite, and clearing the body-determinant of all factors, and putting the arbitraries equal to zero after differentiation, remembering that the composite in this case can contain no odd derivative forms, we have

$$\begin{vmatrix} fx & , 1 , & e^{-ax^2}, & \frac{1}{2^2}e^{-a2^2x^2}, & \ldots , & \frac{1}{n^2}e^{-an^2x^2}, & \frac{1}{(n+1)^2}e^{-a(n+1)^2x^2} \\ fy & , 1 , & e^{-ay^2}, & \frac{1}{2^2}e^{-a2^2y^2}, & \ldots , & \frac{1}{n^2}e^{-an^2y^2}, & \frac{1}{(n+1)^2}e^{-a(n+1)^2y^2} \\ \frac{-1!}{a2!}f''0 & , 0 , & 1 , & 1 & , \ldots , & 1 & , \quad 1 \\ \frac{+2!}{a^2 4!}f^{\text{iv}}0 & , 0 , & 1 , & 2^2 & , \ldots , & n^2 & , \quad (n+1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{(-1)^{n+1} n!}{a^n (2n)!}f^{2n}0, & 0 , & 1 , & 2^{2n-2} & , \ldots , & n^{2n-2} & , \quad (n+1)^{2n-2} \\ \Phi(u) & , 0 , & 0 , & 0 & , \ldots , & 0 & , \quad 1 \end{vmatrix} = 0 . \quad (149)$$

Whence follows

$$fx - fy = \sum_{r=1}^{r=n} A_r \left(e^{-ar^2x^2} - e^{-ar^2y^2}\right) + R \qquad (150)$$

wherein when $n = \infty$, we have

$$A_r = 2 \sum_{p=1}^{p=\infty} \frac{(-1)^r}{r^2 a^p} {}^r\mathfrak{F}_p \frac{p!}{(2p)!} f^{2p}(0) .$$

If $y = 0$, then

$$fx = A_0 + \sum_{r=1}^{r=n} A_r e^{-ar^2x^2} + R , \qquad (151)$$

in which the value of A_0, for $n = \infty$, is obvious.

If $y = \infty$, then

$$\begin{aligned} fx - f\infty &= 2 \sum_{r=1}^{\infty} e^{-ar^2x^2} \sum_{p=1}^{\infty} \frac{(-1)^r}{r^2 a^p} {}^r\mathfrak{F}_p \frac{p!}{(2p)!} f^{2p}(0) \\ &= \int_{\infty}^{x} f'x dx ; \end{aligned}$$

whence

$$\int_x^{\infty} \psi x dx = 2 \sum_{r=1}^{\infty} e^{-ar^2x^2} \sum_{p=1}^{\infty} \frac{(-1)^{r+1}}{r^2 a^p} {}^r\mathfrak{F}_p \frac{p!}{(2p)!} \psi^{2p-1}(0) ,$$

and if $x = 0$,

$$\int_0^{\infty} \psi x dx = \sum_0^{\infty} \frac{1}{a^p (2p+1)!} \frac{p!}{(2p)!} \psi^{2p-1}(0) ,$$

provided fx be a function developable as above.

In (150) put $a = ib$, and

$$e^{-ibr^2x^2} = \cos br^2x^2 - i \sin br^2x^2 .$$

Then

$$fx = A_0 + iB_0 + \sum_{r=1}^{n} (\cos br^2x^2 - i \sin br^2x^2)(A_r + iB_r) + R ,$$

wherein, when $n = \infty$,

$$A_0 = \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} \frac{(-1)^p}{b^{2p}(4p+1)!} \frac{(2p)!}{(4p)!} f^{4p}(0) ,$$

$$B_0 = \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{b^{2p-1}(4p-1)!} \frac{(2p-1)!}{(4p-2)!} f^{4p-2}(0) ,$$

$$A_r = 2 \sum_{p=1}^{\infty} \frac{(-1)^r (-1)^p}{b^{2p}r^2} \frac{(2p)!}{(4p)!} {}^r\mathfrak{F}_{2p} f^{4p}(0) ,$$

$$B_r = 2 \sum_{p=1}^{\infty} \frac{(-1)^r (-1)^{p+1}}{b^{2p-1}r^2} \frac{(2p-1)!}{(4p-2)!} {}^r\mathfrak{F}_{2p-1} f^{4p-2}(0) .$$

Equating real and unreal parts, we have, if fx is real,

$$fx = A_0 + \sum_{1}^{n} (A_r \cos br^2x^2 + B_r \sin br^2x^2) + R , \qquad (152)$$

$$0 = B_0 + \sum_{1}^{n} (B_r \cos br^2x^2 - A_r \sin br^2x^2) + R .$$

41. Consider the function x^2a^x, we have

$$\frac{d^n}{dx^n}(x^2a^x) = [x^2c^n + 2nxc^{n-1} + n(n-1)c^{n-2}]a^x ,$$

wherein $c = \log a$ and $a = e^c$.

If $x = 0$, then

$$\left[\frac{d^n}{dx^n}(x^2a^x)\right]_0 = n(n-1)c^{n-2} .$$

Let $c = -ar^2 = \log a$, so that $a = e^{-ar^2}$. Consider the expansion of fx in terms of the functions

$$x^2 e^{-ar^2x} ,$$

which for brevity in printing the determinant we symbolize by $\varphi_x(r)$.

Noticing that these functions do not contain the first power of x, it follows that the composite will not contain the first derivative row. Making the substitutions and factoring the body-determinant to its simplest form, after putting the arbitraries equal to zero after differentiation and observing the

formula

$$\left[\frac{d^n}{dx^n}(x^2e^{-ar^2x})\right]_0 = (-1)^{n-2}\,n(n-1)\,a^{n-2}\,r^{2n-4},$$

we have

$$\begin{vmatrix} fx & ,1, & \varphi_x(1), & \varphi_x(2), & \ldots, & \varphi_x(n), & \varphi_x(n+1) \\ fy & ,1, & \varphi_y(1), & \varphi_y(2), & \ldots, & \varphi_y(n), & \varphi_y(n+1) \\ +\frac{1}{1\,.\,2}f''0 & ,0, & 1, & 1, & \ldots, & 1, & 1 \\ -\frac{1}{a}\frac{1}{2\,.\,3}f'''0 & ,0, & 1, & 2^2, & \ldots, & n^2, & (n+1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{(-1)^n}{a^{n-2}n(n-1)}f^n0, & 0, & 1, & 2^{2n-4}, & \ldots, & n^{2n-4}, & (n+1)^{2n-4} \\ \Phi(u) & ,0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0. \qquad (153)$$

Expanding this we get the formula

$$fx - fy = \sum_{r=1}^{n} A_r\,(x^2e^{-ar^2x} - y^2e^{-ar^2y}) + R\,, \qquad (154)$$

wherein, when $n = \infty$,

$$A_r = 2\sum_{p=2}^{\infty}\frac{(-1)^{r+1}}{a^{p-2}p(p-1)}\,{}^r\mathfrak{F}_{p-1}\,f^p(0)\,.$$

If we put $y = 0$ and $a = ib$ in (154), we have, after equating real parts, if fx is a real function,

$$fx = f0 + x^2\sum_{r=1}^{n}(A'_r\cos br^2x + B'_r\sin br^2x) + R\,, \qquad (155)$$

wherein, when $n = \infty$,

$$A'_r = 2\sum_{p=1}^{\infty}\frac{(-1)^{r+1}(-1)^{p+1}}{a^{2p-2}\,2p(2p-1)}\,{}^r\mathfrak{F}_{2p-1}\,f^{2p}(0)\,,$$

$$B'_r = 2\sum_{p=1}^{\infty}\frac{(-1)^{r+1}(-1)^{p}}{a^{2p-1}\,2p(2p+1)}\,{}^r\mathfrak{F}_{2p}\,f^{2p+1}(0)\,.$$

42. Before leaving these examples of exponential forms, let us return to the form

$$fx = \sum_{r=0}^{\infty} A_r\,e^{rax}\,,$$

when the exponent of e is a real quantity. We have seen how any expansion in powers of e leads to an expansion of a complex harmonic function, but none of these complex harmonics got from exponential series have led to Fourier's

form, and only the expansion above can lead to that well established theorem. The expansion must therefore be possible, and in casting about for an explanation of the seeming difficulty I have sought satisfaction in the following course of reasoning.*

Consider the expansion of fx according to the functions e^{-rax}, in which a and x are positive and r successively the integers 1, 2, 3, ...

If this expansion be possible, then we must have

$$fx = \sum_0^\infty A_r\, e^{-rax}\,,$$

wherein the coefficients A_r are independent of x. This being so, we have, after n differentiations,

$$f^n x = \sum_1^\infty A_r\,(-1)^n\, r^n a^n e^{-rax}\,.$$

Therefore, since the second member vanishes when x is infinite, we must have $f^n\infty = 0$. Hence we observe that any one of the derivative rows in the composite may be made to vanish, and therefore the determinant, by putting $x = \infty$ after differentiation. Hence, by exactly the same method of reasoning as that employed in establishing the theorem of § 4, it follows that we may omit, in this case, from the composite any derivative row, provided we include infinity among the arbitraries, between the greatest and least of which lies u. We may, therefore, omit all of the odd derivative rows or all of the even derivative rows. But at no time can two consecutive rows be omitted.

Making the substitutions in the composite, containing only even derivative rows, and putting the arbitraries after differentiation equal to l, after factoring to simplest form, we have

$$\begin{vmatrix} fx & , 1, & e^{a(l-x)}, & \frac{1}{2^2}e^{2a(l-x)}, & \ldots, & \frac{1}{n^2}e^{na(l-x)}, & \frac{1}{(n+1)^2}e^{(n+1)a(l-x)} \\ f(l) & , 1, & 1\;, & \frac{1}{2^2}\;, & \ldots, & \frac{1}{n^2}\;, & \frac{1}{(n+1)^2} \\ \frac{1}{a^2}f''(l) & , 0, & 1\;, & 1\;, & \ldots, & 1\;, & 1 \\ \frac{1}{a^4}f^{\text{iv}}(l) & , 0, & 1\;, & 2^2\;, & \ldots, & n^2\;, & (n+1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{a^{2n}}f^{2n}(l), & 0, & 1\;, & 2^{2n-2}\;, & \ldots, & n^{2n-2}\;, & (n+1)^{2n-2} \\ \Phi(u) & , 0, & 0\;, & 0\;, & \ldots, & 0\;, & 1 \end{vmatrix} = 0\,. \quad (156)$$

* Which is far from being satisfactory. June, 1893.

So that

$$fx = A_0 + \sum_{r=1}^{n}{}' A_r\, e^{ra(l-x)} + R\,, \tag{157}$$

wherein, for $n = \infty$,

$$A_0 = f(l) - 2 \sum_{r=1}^{r=\infty}{}' \sum_{p=1}^{p=\infty}{}' \frac{(-1)^{r+1}(-1)^{p+1}}{r^2 a^{2p}}\, {}^r\mathfrak{F}_p\, f^{2p}(l)\,,$$

$$A_r = 2 \sum_{p=1}^{\infty}{}' \frac{(-1)^{r+1}(-1)^{p+1}}{r^2 a^{2p}}\, {}^r\mathfrak{F}_p\, f^{2p}(l)\,.$$

In like manner we deduce from the composite in which we retain the odd derivative rows,

$$fx = B_0 + \sum_{r=1}^{n}{}' B_r e^{ra(l-x)} + R\,, \tag{158}$$

in which, for $n = \infty$,

$$B_0 = fl - 2 \sum_{r=1}^{\infty}{}' \sum_{p=1}^{\infty}{}' \frac{(-1)^{r+1}(-1)^{p}}{a^{2p-1} r}\, {}^r\mathfrak{F}_p\, f^{2p-1}(l)\,,$$

$$B_r = 2 \sum_{p=1}^{\infty}{}' \frac{(-1)^{r+1}(-1)^{p}}{r a^{2p-1}}\, {}^r\mathfrak{F}_p\, f^{2p-1}(l)\,.$$

If in (157) and (158) we put $a = ib$, and $b = 2\pi/l$, also

$$e^{rib(l-x)} = \cos rb(l-x) + i \sin rb(l-x)\,,$$

we have by equating reals in (157),

$$fx = A'_0 + \sum_{r=1}^{\infty}{}' A'_r \cos \frac{2\pi r}{l} x\,, \tag{159}$$

wherein

$$A'_0 = \frac{1}{l} \int_{\frac{1}{2}l}^{\frac{3}{2}l} fx dx\,, \quad A'_r = 2 \sum_{p=1}^{\infty}{}' \frac{(-1)^{r+1}}{r^2}\, {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p}(l)\,,$$

which is the same as (134).

Equating imaginaries in (158) we have

$$fx = fl + \sum_{r=1}^{\infty}{}' B'_r \sin \frac{2\pi r}{l} x\,,$$

wherein

$$B'_r = 2 \sum_{p=1}^{\infty}{}' \frac{(-1)^{r+1}}{r}\, {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p-1} f^{2p-1}(l)\,,$$

which is the same form as (138).

Hyperbolic Functions.

43. If it be possible to expand a function in terms of the hyperbolic sines and cosines, we may obtain the form of that expansion in a manner almost identical with that employed for the circular functions.

Thus, consider the form of the expansion in terms of the functions

$$\frac{1}{r^2}\cosh rax\,.$$

Putting $x = z$ after differentiation, and observing that the odd rows are to be left out, because cosh is an even function, we deduce from the composite

$$fx - fy = \sum_{r=1}^{n}{}' \frac{A_r}{\cosh raz}(\cosh rax - \cosh ray) + R\,, \tag{160}$$

wherein, if $n = \infty$,

$$A_r = 2\sum_{p=1}^{\infty}{}' \frac{(-1)^{r+1}(-1)^{p+1}}{a^{2p}r^2}\,{}^r\mathfrak{F}_p f^{2p}z\,.$$

Put $y = z = 0$; then, if $a = 1/l$,

$$fx = A'_0 + \sum_{r=1}^{n} A'_r \cosh\frac{rx}{l} + R\,, \tag{161}$$

wherein, if infinite values of n be permissible,

$$A'_0 = \sum_{0}^{\infty}\frac{(-1)^p\,l^{2p}}{(2p+1)!}f^{2p}(0)\,, \qquad A'_r = \sum_{p=1}^{\infty}\frac{(-1)^{r+1}}{r^2}(-1)^{p+1}\,l^{2p}\,{}^r\mathfrak{F}_p f^{2p}(0).$$

Again, in (160) put $a = i\pi/l$ and $z = y = l$, then $\cosh ri\pi = (-1)^r$ and $a^{2p} = (-1)^p(\pi/l)^{2p}$;

$$\therefore \qquad fx = A_0'' + \sum_{r=1}^{n}{}' A_r'' \cosh\frac{ri\pi}{l}x + R\,, \tag{162}$$

and

$$A_0'' = \frac{1}{l}\int_{\frac{1}{2}l}^{\frac{3}{2}l} fx\,dx\,, \qquad A_r'' = 2\sum_{p=1}^{\infty}\frac{1}{r^2}\,{}^r\mathfrak{F}_p\left[\frac{l}{2\pi}\right]^{2p} f^{2p}(l)\,.$$

In exactly the same manner we deduce the formula

$$fx - fy = \sum_{r=1}^{n}{}' \frac{B_r}{\cosh raz}(\sinh rax - \sinh ray) + R\,, \tag{163}$$

wherein, if $n = \infty$,

$$B_r = 2\sum_{p=1}^{\infty}{}' \frac{(-1)^{r+1}(-1)^{p+1}}{ra^{2p-1}}\,{}^r\mathfrak{F}_p\, f^{2p-1}z\,.$$

From which we deduce a formula similar to (161) by putting $y = z = 0$ and $a = 1/l$.

Thus

$$fx = f(0) + \sum_{r=1}^{n} B_r' \sinh \frac{rx}{l} + R. \tag{164}$$

If fx is a function which vanishes with x, then $f(0) = 0$. By addition of (161) and (164), we have

$$fx = C_0 + \sum_{r=1}^{n} \left[C_r' \cosh \frac{rx}{l} + C_r'' \sinh \frac{rx}{l} \right] + R, \tag{165}$$

a formula analagous to Fourier's theorem, in which the values of the constant coefficients. when $n = \infty$, are at once obvious from the above.

Bessel's Functions.

44. Consider the expansion of fx in terms of the functions

$$J_{2m}(rax) = \sum_{l=0}^{\infty} \frac{(-1)^l (ra)^{2m+2l}}{l!\,(2m+l)!} \left[\frac{x}{2}\right] \quad ,$$

in which m is any positive integer including zero. This function is an even function containing no power of x lower than $2m$. The composite will, by § 4, contain no odd derivative rows, nor will it contain any even derivative row of lower order than $2m$.

Substitute in the composite the functions

$$\frac{1}{r^{2m}} J_{2m}(rax),$$

and put $x = 0$ after differentiation. Factoring to simplest shape, we have

$$\begin{vmatrix}
 & fx, & 1, & J_{2m}(ax), & \frac{1}{2^{2m}} J_{2m}(2ax), & \ldots, & \frac{1}{n^{2m}} J_{2m}(nax), & \frac{1}{(n+1)^{2m}} J_{2m}(\overline{n+1}\,ax) \\
 & fy, & 1, & J_{2m}(ay), & \frac{1}{2^{2m}} J_{2m}(2ay), & \ldots, & \frac{1}{n^{2m}} J_{2m}(nay), & \frac{1}{(n+1)^{2m}} J_{2m}(\overline{n+1}\,ay) \\
\left[\frac{2}{a}\right]^{2m} & f^{2m}(0), & 0, & 1, & 1, & \ldots, & 1, & 1 \\
-\frac{1!\,(2m+1)!}{(2m+2)!}\left[\frac{2}{a}\right]^{2m+2} & f^{2m+2}(0), & 0, & 1, & 2^2, & \ldots, & n^2, & (n+1)^2 \\
+\frac{2!\,(2m+2)!}{(2m+4)!}\left[\frac{2}{a}\right]^{2m+4} & f^{2m+4}(0), & 0, & 1, & 2^4, & \ldots, & n^4, & (n+1)^4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(-1)^{n-1}\frac{(n-1)!\,(2m+n-1)!}{[2m+2(n-1)]!}\left[\frac{2}{a}\right]^{2(m+n-1)} & f^{2(m+n-1)}(0), & 0, & 1, & 2^{2n-2}, & \ldots, & n^{2n-2}, & (n+1)^{2n-2} \\
 & \Phi(u), & 0, & 0, & 0, & \ldots, & 0, & 1
\end{vmatrix} = 0. \tag{167}$$

Expanding this we obtain

$$fx - fy = \sum_{r=1}^{n}{}' \frac{(-1)^{r+1}}{r^{2m}} A_r \left[J_{2m}(rax) - J_{2m}(ray)\right] + R \,, \tag{168}$$

wherein, if $n = \infty$,

$$A_r = 2 \sum_{p=1}^{\infty}{}' \frac{(p-1)!\,(2m+p-1)!}{(2m+2p-1)!} \left[\frac{2}{a}\right]^{2(m+p-1)} {}^r\mathfrak{J}_p\, f^{2(m+p-1)}(0) \,.$$

If $y = 0$ and fx a function vanishing with x, then for $m > 0$ we have

$$fx = \sum_{r=1}^{n}{}' \frac{(-1)^{r+1}}{r^{2m}} A_r\, J_{2m}(rax) + R \,. \tag{169}$$

In exactly the same manner we deduce the expansion in terms of any odd function $J_{2m-1}(rax)$, there being no difference in the result, save that we must change the even derivatives to odd ones.

So that we have

$$fx - fy = \sum_{r=1}^{n}{}' A_r \left[J_m(rax) - J_m(ray)\right] \,, \tag{170}$$

wherein, if $n = \infty$,

$$A_r = 2 \sum_{p=1}^{\infty}{}' \frac{(p-1)!\,(m+p-1)!}{(m+2p-1)!} \left[\frac{2}{a}\right]^{m+2(p-1)} \frac{(-1)^{r+1}}{r^m} {}^r\mathfrak{J}_p\, f^{m+2(p-1)}(0) \,.$$

If $m > 0$ there will be no absolute term, since $J_0(ray)$ is the only function which has a constant term when we put $y = 0$.

In (170) put $m = 0$ and $a = 2$, also $y = 0$; then, since $J_0(ray) = -1$, we have

$$fx = B_0 + \sum_{r=1}^{n}{}' B_r\, J_0(2rx) + R \,, \tag{171}$$

wherein, for $n = \infty$,

$$B_0 = f(0) + 2 \sum_{r=1}^{\infty}{}' \sum_{p=1}^{\infty}{}' (-1)^r \frac{(p\,!)^2}{(2p)!} {}^r\mathfrak{J}_p f^{2p}(0) \,,$$

$$B_r = 2 \sum_{p=1}^{\infty}{}' (-1)^{r+1} \frac{(p\,!)^2}{(2p)!} {}^r\mathfrak{J}_p\, f^{2p}(0) \,.$$

This last result (171) is the remarkable theorem due to Schlömilch (Tod-

hunter's Functions, p. 336). The value of the coefficient as determined by Schlömilch, however, is

$$B_r = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\left\{ f(0) + u\int_0^1 \frac{f'(u\xi)\,d\xi}{\sqrt{(1-\xi^2)}}\right\}\cos 2rudu\,,$$

for every value of u except zero, when we must add $2f(0)$.

Other interesting forms may be obtained from (170) through suppositions regarding the arbitrary constants.

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